Young Mathematicians Conference 2019
The Ohio State University, August 9-11

When bands play in Random Matrix Theory

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Abstract of Report Talk: One of the main interests of Random Matrix Theory is the study of the distribution of eigenvalues of matrices under different symmetry constraints. Imposed structures lead to different behavior for the limiting distribution, which can be connected to problems in diverse fields. Certain structured random matrices, for instance, have been successfully applied to model zeros of L-functions and energy levels of heavy nuclei. The connection to Nuclear Physics blossomed in the 1950's with the work of Wigner and others; connections with Number Theory were found two decades later through Montgomery's pair correlation conjecture for the Riemann zeta function.
There is now a large body of work on various symmetries of matrices. While the eigenvalue density of the family of real symmetric matrices converges to a semicircle, different behavior emerges as the symmetry increases. Two different groups completely analyzed the case of real symmetric Toeplitz matrices in 2005, seeing a new distribution that is almost Gaussian; this was extended in 2007 to real symmetric palindromic Toeplitz matrices – so the first row of the matrix is a palindrome – where the extra symmetry leads to Gaussian behavior. We study a new case: $N \times N$ symmetric palindromic Toeplitz matrices constructed by adding constant diagonals, called bands, to the center and corners of the zero matrix.

$$A_{N,2} = \begin{bmatrix} 0 & x_1 & x_2 & 0 & \dots & 0 & x_2 & x_1 & 0 \\ x_1 & 0 & x_1 & x_2 & 0 & & 0 & x_2 & x_1 \\ x_2 & x_1 & 0 & x_1 & x_2 & \ddots & & 0 & x_2 \\ 0 & x_2 & x_1 & \ddots & \ddots & \ddots & 0 & & 0 \\ \vdots & 0 & x_2 & \ddots & \ddots & \ddots & x_2 & 0 & \vdots \\ 0 & & 0 & \ddots & \ddots & \ddots & x_1 & x_2 & 0 \\ x_2 & 0 & & \ddots & x_2 & x_1 & 0 & x_1 & x_2 \\ x_1 & x_2 & 0 & & 0 & x_2 & x_1 & 0 & x_1 \\ 0 & x_1 & x_2 & 0 & \dots & 0 & x_2 & x_1 & 0 \end{bmatrix}$$

If the matrix $A_{N,D}$ has D bands, then it will have non-zero entries on the D diagonals above and below the main diagonal, plus the corresponding upper and lower parts due to the palindromic constraint. Banded matrices have a long history; band one matrices – not constant on diagonals – are related to the Laplacian of some systems in mathematical physics. Further, the behavior of their eigenvalues transitions from that of the density they are drawn from, if we have diagonal matrices, to the semi-circle when $D = N$, with the transition to semi-circular behavior essentially complete once $D \approx \sqrt{N}$.

The imposed additional symmetry of the palindromic Toeplitz case greatly increases the number of terms that contribute to the moments of the eigenvalue distribution compared to the case of band matrices which are not constant along diagonals. Using the method of moments, we reduce the problem of determining the density of eigenvalues to some combinatorial problems. We derive and prove formulas for these values, which we then use to prove corresponding formulas for the density of eigenvalues and derive convergence results on how rapidly it approaches Gaussian as the number of bands increases.

Received: July 21, 2019